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Noether Theorem in Higher-Order Lagrangian Mechanics

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We study higher-order Lagrangian mechanics on the k-velocity manifold. The variational problem gives rise to new concepts, such as main invariants, Zermelo conditions, higher-order energies, and new conservation laws. A theorem of Noether type is proved for higher-order Lagrangians. The invariants to the infinitesimal symmetries are explicitly written. All this construction is a natural extension of classical Lagrangian mechanics.

Analytical mechanics based on Lagrangians defined on higher-order jet spaces has been studied with remarkable results by many people (Andreas *et al.*, 1991; Craig, 1935, Crampin *et al.*, 1986; Grigore, 1993; Kawaguchi, 1961; Kondo, 1991; Krupka, 1983; Krupkova, 1992; Leon *et al.*, 1985; Mangiarotti and Modugno, 1982; Miron and Atanasiu, 1994; Saunders, 1989; Synge, 1935; Yano and Ishihara, 1973).

The Lagrangian formalism is based on the so-called Poincaré–Cartan 1-form (Garcia, 1974; Grigore, 1993; Crampin *et al.*, 1986), or, more naturally, on a 2-form having as associated system the Euler–Lagrange equations (Crampin *et al.*, 1986; Grigore, 1993; Krupka, 1983).

A problem of interest is how to study the Lagrangians defined on the higher-order velocity space by methods which are straightforward extensions of the classical ones. More precisely, how can one derive the Euler–Lagrange equations from the condition that the integral of action

$$I(c) = \int_0^1 L\left(x, \frac{dx}{dt}, \ldots, \frac{1}{k!}\frac{d^kx}{dt^k}\right) dt \tag{I}$$

satisfies the Hamilton principle or prove a Noether theorem in the classical manner? Of course, such a development of higher-order analytical mechanics does not have the same generality as that which is based on the Poincaré–

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Cartan–Sourieau formalism. But it has a great advantage: it would be a natural extension of the original Lagrangian classical mechanics.

The aim of the present paper is to get such an extension of classical Lagrangian mechanics to higher-order Lagrangian mechanics.

For simplicity we study here only autonomous Lagrangians. The nonautonomous case will be treated in the same manner in a forthcoming paper.

We begin with some preliminary considerations about the geometry of the total space of the k-jet bundle (or k-velocity bundle) $J_0^k M$. Here $J_0^k M$ (Crampin *et al.*, 1986; Miron and Atanasiu, n.d.-a) is identified with the k-osculator bundle $\mathrm{Osc}^k M$, which has an obvious geometrical meaning. On the manifold $E = \mathrm{Osc}^k M$ we introduce below in (1.3) the Liouville vector fields $\overset{(1)}{\Gamma}, \ldots, \overset{(k)}{\Gamma}$, the k-tangent structure J in (1.4), and the notion of the nonlinear connection N. These lead to the direct decomposition (1.5), which is essential in the geometry of the k-osculator bundle (Miron and Atanasiu, n.d.-a).

In Section 2 we define the differentiable higher-order Lagrangian $L(x, y^{(1)}, \ldots, y^{(k)})$ on E and remark on the case when L is regular. The Lie derivatives with respect to the Liouville vector fields $\overset{(1)}{\Gamma}, \ldots, \overset{(k)}{\Gamma}$ of a Lagrangian L determine the main invariants $I^1(L), \ldots, I^k(L)$. Theorem 2.1 gives the necessary conditions (2.8), called Zermelo conditions, in order for the integral of action I not to depend on the parametrization of the curve c.

In Section 3, following some ideas of Synge (1935), we present the variational problem for the integral of action $I(c)$. Now we introduce new invariants $I_V^1(L), \ldots, I_V^k(L)$, establish the identity (3.5), and the Euler–Lagrange equations (3.7). Of course we show that $\overset{0}{E}_i(L)$ is a covector field.

Also, the variational problem suggests (in Section 4) that we define in (4.2) the operator d_V/dt and state the relations between the operators d_V/dt, $I_V^1, \ldots, I_V^k$. All these operators are extremely useful in the formulation of Noether theorem.

Section 5 is devoted to the so-called Craig–Synge covectors $\overset{1}{E}_i(L), \ldots, \overset{k}{E}_i(L)$. Theorems 5.2 and 5.3 give the main properties of these covectors.

Section 6 is dedicated to the notions of energy and to the higher-order energies $\mathscr{E}(L), \mathscr{E}_c^1(L), \ldots, \mathscr{E}_c^k(L)$. Theorem 6.1 affirms the existence for $k > 1$ of some obstructions to the conservation of the energy $\mathscr{E}(L)$ along of the solution curves of the Euler–Lagrange equations $\overset{0}{E}_i(L) = 0$. Formula (6.8) is remarkable. It has as a consequence (Theorem 6.3) that the energy of order k, $\mathscr{E}_c^k(L)$, is conserved along the mentioned curves.

Remarking that the energy $\mathscr{E}(L)$ is globally defined on $E = \mathrm{Osc}^k M$, but it does not satisfy the conservation law, it is expected that a Noether theorem referring to the autonomous Lagrangians $L(x, y^{(1)}, \ldots, y^{(k)})$, $k \geq 1$, will include essentially the energies of higher order $\mathscr{E}_c^1(L), \ldots, \mathscr{E}_c^k(L)$.

In Section 7 we define the notion of symmetry and prove a Noether-type theorem. It states that, with respect to an infinitesimal symmetry (7.1), the scalar functions $\mathscr{F}^k(L, \phi)$ in (7.11) are conserved along the solution curves of the Euler–Lagrange equations $\overset{0}{E}_i(L) = 0$.

In the particular case $k = 1$ or $k = 2$ the functions $\mathscr{F}^k(L, \phi)$ have a remarkable form. And if the Lagrangian L satisfies the Zermelo conditions, then the functions $\mathscr{F}^k(L, \phi)$ are given in Theorem 7.3.

A first application to the higher-order electrodynamics described by the Lagrangian (2.3) can be worked out.

A final remark. The previous theory is very useful in the higher-order Lagrange geometry (Miron and Atanasiu, n.d.-b) based on regular Lagrangians of the form $L(x, y^{(1)}, \ldots, y^{(k)})$. In this geometry a general gauge theory could be obtained following the present results and also those due to Asanov (1985), Crampin *et al.* (1986), Sarlet *et al.* (1987), Krupka (1983), Garcia (1974), Grigore (1993), Andreas *et al.* (1991), Leon and Marrero (1991), Leon *et al.* (1985, 1992), Saunders (1989), Souriau (1970), and many others.

1. PRELIMINARIES

Let M be a real, n-dimensional C^∞-manifold and $(J_0^k M, \pi, M)$ its k-jet bundle (or k-velocity bundle). It will be identified with a k-osculator bundle $(\mathrm{Osc}^k M, \pi, M)$, in which each point $u \in \mathrm{Osc}^k M$ is considered to be a "k-osculator space" of the manifold M at the point $x_0 = \pi(u)$. Namely, if $c: I \to M$ is a smooth curve whose image belongs to a domain U of a local chart on M, $x_0 \in c$, and c is represented by the equations $x^i = x^i(t)$, $t \in I$, $0 \in I$, $x_0 = (x^i(0))$, then the point $u \in \mathrm{Osc}^k M$ can be represented by a small arc of a curve given by

$$x^{*i}(t) = x^i(0) + t\frac{dx^i}{dt}(0) + \cdots + \frac{1}{k!}t^k\frac{d^k x^i}{dt^k}(0), \qquad t \in (-\epsilon, \epsilon) \subset I$$

The indexes $i, j, h, r, s, \ldots$ run over the set $\{1, 2, \ldots, n\}$.

Therefore, on $\pi^{-1}(U)$, the coordinates of the point $u \in \mathrm{Osc}^k M$ can be given in the form

$$x^i = x^i(0), \qquad y^{(\alpha)i} = \frac{1}{\alpha!}\frac{d^\alpha x^i}{dt^\alpha}(0), \qquad \alpha = 1, \ldots, k \tag{1.0}$$

We write $E = \mathrm{Osc}^k M$ and notice that by means of (1.0) the following local coordinate transformations $(x^i, y^{(1)i}, \ldots, y^{(k)i}) \to (\tilde{x}^i, \tilde{y}^{(1)i}, \ldots, \tilde{y}^{(k)i})$

are obtained:

$$\tilde{x}^i = \tilde{x}^i(x^1, \ldots, x^n), \qquad \operatorname{rank}\|\partial\tilde{x}^i/\partial x^j\| = n$$

$$\tilde{y}^{(1)i} = \frac{\partial\tilde{x}^i}{\partial x^j} y^{(1)j}$$

$$2\tilde{y}^{(2)i} = \frac{\partial\tilde{y}^{(1)i}}{\partial x^j} y^{(1)j} + 2\frac{\partial\tilde{y}^{(1)i}}{\partial y^{(1)j}} y^{(2)j}$$

$$\vdots$$

$$k\tilde{y}^{(k)i} = \frac{\partial\tilde{y}^{(k-1)i}}{\partial x^j} y^{(1)j} + 2\frac{\partial\tilde{y}^{(k-1)i}}{\partial y^{(1)j}} y^{(2)j} + \cdots + k\frac{\partial\tilde{y}^{(k-1)i}}{\partial y^{(k-1)j}} y^{(k)j} \tag{1.1}$$

We have

$$\frac{\partial\tilde{x}^i}{\partial x^j} = \frac{\partial\tilde{y}^{(1)i}}{\partial y^{(1)j}} = \cdots = \frac{\partial\tilde{y}^{(k)i}}{\partial y^{(k)j}}$$

$$\frac{\partial\tilde{y}^{(1)i}}{\partial x^j} = \frac{\partial\tilde{y}^{(2)i}}{\partial y^{(1)j}} = \cdots = \frac{\partial\tilde{y}^{(k)i}}{\partial y^{(k-1)j}}; \quad \text{etc.}$$

The simple form (1.1) allows us to check the geometrical character of the notions which we use in this paper. For instance, $\operatorname{rank}\|y^{(1)i}\| = 1$ has a geometrical character. Then

$$\tilde{E} = \{u \in E \,|\, u = (x^i, y^{(1)i}, \ldots, y^{(k)i}), \operatorname{rank}\|y^{(1)i}\| = 1\}$$

is an open submanifold in E. This is an important fact in our construction of the higher-order Lagrange geometry.

Looking for transformations of the natural basis $(\partial/\partial x^i, \partial/\partial y^{(1)i}, \ldots, \partial/\partial y^{(k)i})$ with respect to (1.1), given by

$$\frac{\partial}{\partial x^i} = \frac{\partial\tilde{y}^{(k)j}}{\partial x^i}\frac{\partial}{\partial\tilde{y}^{(k)j}} + \cdots + \frac{\partial\tilde{y}^{(1)j}}{\partial x^i}\frac{\partial}{\partial\tilde{y}^{(1)j}} + \frac{\partial\tilde{x}^j}{\partial x^i}\frac{\partial}{\partial\tilde{x}^j}$$

$$\frac{\partial}{\partial y^{(1)i}} = \frac{\partial\tilde{y}^{(k)j}}{\partial y^{(1)i}}\frac{\partial}{\partial\tilde{y}^{(k)j}} + \cdots + \frac{\partial\tilde{y}^{(1)j}}{\partial y^{(1)i}}\frac{\partial}{\partial\tilde{y}^{(1)j}}$$

$$\vdots$$

$$\frac{\partial}{\partial y^{(k)i}} = \frac{\partial\tilde{y}^{(k)j}}{\partial y^{(k)i}}\frac{\partial}{\partial\tilde{y}^{(k)j}} \tag{1.2}$$

we see that the vector fields $\{\partial/\partial y^{(k)i}\}$ generate a distribution V_k on E of the local dimension n; $\{\partial/\partial y^{(k-1)i}, \partial/\partial y^{(k)i}\}$ determine a distribution V_{k-1} on E of local dimension $2n$; . . . ; $\{\partial/\partial y^{(1)i}, \ldots, \partial/\partial y^{(k)i}\}$ give the vertical distribution $V = V_1$ on E of local distribution kn. We have

$$V_1 \supset V_2 \supset \cdots \supset V_k$$

By means of (1.1) we can prove that

$$\overset{(1)}{\Gamma} = y^{(1)i} \frac{\partial}{\partial y^{(k)i}}$$

$$\overset{(2)}{\Gamma} = y^{(1)i} \frac{\partial}{\partial y^{(k-1)i}} + 2y^{(2)i} \frac{\partial}{\partial y^{(k)i}}$$

$$\vdots$$

$$\overset{(k)}{\Gamma} = y^{(1)i} \frac{\partial}{\partial y^{(1)i}} + 2y^{(2)i} \frac{\partial}{\partial y^{(2)i}} + \cdots + ky^{(k)i} \frac{\partial}{\partial y^{(k)i}} \tag{1.3}$$

are global vector fields on E. Here $\overset{(1)}{\Gamma}$ belongs to the distribution V_k, $\overset{(2)}{\Gamma}$ belongs to the distribution $V_{k-1}, \ldots, \overset{(k)}{\Gamma}$ belongs to the distribution V_1. They are called *the Liouville vector fields* on E. For $k = 1$, $\overset{(1)}{\Gamma}$ is the classical Liouville vector field on the tangent bundle TM of the manifold M.

The existence of the distribution $V_1, \ldots, V_k$ allows us to introduce the notion of a k-tangent structure J: $\mathcal{X}(E) \to \mathcal{X}(E)$ defined by

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^{(1)i}}, \qquad J\left(\frac{\partial}{\partial y^{(1)i}}\right) = \frac{\partial}{\partial y^{(2)i}}, \ldots,$$

$$J\left(\frac{\partial}{\partial y^{(k-1)i}}\right) = \frac{\partial}{\partial y^{(k)i}}, \qquad J\left(\frac{\partial}{\partial y^{(k)i}}\right) = 0 \tag{1.4}$$

We get

Theorem 1.1. The k-tangent structure J has the following properties:

1. J is globally defined on E.
2. $\operatorname{rank}\|J\| = kn$; $\overbrace{J^0 J^0 \cdots {}^0 J}^{k+1} = 0$.
3. $J(\overset{(k)}{\Gamma}) = \overset{(k-1)}{\Gamma}, \ldots, J(\overset{(2)}{\Gamma}) = \overset{(1)}{\Gamma}, J(\overset{(1)}{\Gamma}) = 0$.
4. J is an integrable structure.

A k-spray on E is a vector field S on E such that $JS = \overset{(k)}{\Gamma}$.

A nonlinear connection is a vectorial subbundle $N(E)$ of the tangent bundle $T(E)$ such that the Whitney sum

$$T(E) = N(E) \oplus V(E)$$

holds.

Putting $N_0 = N$, $N_1 = J(N_0), \ldots, N_{k-1} = J(N_{k-2})$ we obtain the direct decomposition

$$T_u(E) = N_0(u) \oplus N_1(u) \oplus \cdots \oplus N_{k-1}(u) \oplus V_k(u), \forall u \in E \tag{1.5}$$

The geometry of the total space $E = \mathrm{Osc}^k M$ is studied by means of the direct sum (1.5). We have proved that a k-spray determines a nonlinear connection. Other important geometric objects on E such as linear connections, Riemannian metrics, etc., are always expressed by means of the decomposition (1.5).

2. HIGHER-ORDER LAGRANGIANS. THE MAIN INVARIANTS. ZERMELO CONDITIONS

A scalar field $L(x, y^{(1)}, \ldots, y^{(k)})$ on E is called a (an autonomous) higher-order differentiable Lagrangian if it is of C^∞-class on $\tilde{E}$ and continuous at the points $u \in E$ for which $y^{(1)i} = 0$.

Using (1.2), we can prove that

$$g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^{(k)i} \partial y^{(k)j}} \tag{2.1}$$

is a distinguished tensor field (a d-tensor) on $\tilde{E}$. That is, with respect to (1.1) we have

$$\tilde{g}_{ij} = \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} g_{rs}$$

If

$$\mathrm{rank}\|g_{ij}(x, y^{(1)}, \ldots, y^{(k)})\| = n \qquad \text{on} \quad \tilde{E} \tag{2.2}$$

we say that $L(x, y^{(1)}, \ldots, y^{(k)})$ is a regular Lagrangian.

For the moment we study higher-order differentiable Lagrangians without the regularity condition.

For example,

$$L(x, y^{(1)}, \ldots, y^{(k)}) = \gamma_{ij}(x) z^{(k)i} z^{(k)j} + a_i(x, y^{(1)}, \ldots, y^{(k-1)}) z^{(k)i} + b(x, y^{(1)}, \ldots, y^{(k-1)}) \tag{2.3}$$

is a regular Lagrangian of order k (Miron and Atanasiu, n.d.-b). This is a generalization of a very well known Lagrangian from electrodynamics.

For any differentiable Lagrangian $L(x, y^{(1)}, \ldots, y^{(k)})$ we take the Lie derivatives with respect to the Liouville vector fields $\overset{(1)}{\Gamma}, \ldots, \overset{(k)}{\Gamma}$:

$$I^1(L) = \mathcal{L}_\Gamma^{(1)} L, \ldots, I^k(L) = \mathcal{L}_\Gamma^{(k)} L \tag{2.4}$$

The course $I^1(L), \ldots, I^k(L)$ are scalars on E and differentiable on $\tilde{E}$. They are important in the study of higher-order Lagrangians. We say that $I^1(L), \ldots, I^k(L)$ are *the main invariants of* the Lagrangian $L(x, y^{(1)}, \ldots, y^{(k)})$. They have the expressions

$$I^1(L) = y^{(1)i} \frac{\partial L}{\partial y^{(k)i}}, \qquad I^2(L) = y^{(1)i} \frac{\partial L}{\partial y^{(k-1)i}} + 2y^{(2)i} \frac{\partial L}{\partial y^{(k)i}}, \ldots,$$
$$I^k(L) = y^{(1)i} \frac{\partial L}{\partial y^{(1)i}} + 2y^{(2)i} \frac{\partial L}{\partial y^{(2)i}} + \cdots + ky^{(k)i} \frac{\partial L}{\partial y^{(k)i}} \tag{2.5}$$

Let us consider c: $[0, 1] \to M$ a smooth curve, $c(t) = (x^i(t))$, $t \in [0, 1]$. Its extension to the manifold $\tilde{E}$ is

$$c^*: t \in [0, 1] \to \left(x^i(t), \frac{dx^i}{dt}, \ldots, \frac{1}{k!} \frac{d^k x^i}{dt^k}\right) \in \tilde{E} \tag{2.6}$$

The integral of action of $L(x, y^{(1)}, \ldots, y^{(k)})$ on c is defined by

$$I(c) = \int_0^1 L\left(x(t), \frac{dx}{dt}, \ldots, \frac{1}{k!} \frac{d^k x}{dt^k}\right) dt \tag{2.7}$$

Now we can prove:

Theorem 2.1. The necessary conditions that $I(c)$ does not depend on the parametrization of the curve c are

$$I^1(L) = \cdots = I^{k-1}(L) = 0, \qquad I^k(L) = L \tag{2.8}$$

Proof. Let $\tilde{t} = \tilde{t}(t)$, $t \in [0, 1]$, be a differentiable diffeomorphism. In order for the integral of action $I(c)$ not to depend on the parametrization of the curve c is necessary that

$$\tilde{L}\left(\tilde{x}, \frac{d\tilde{x}^i}{d\tilde{t}}, \ldots, \frac{1}{k!} \frac{d^k \tilde{x}^i}{d\tilde{t}^k}\right) \frac{d\tilde{t}}{dt} \tag{*}$$
$$= L\left(x, \frac{dx}{d\tilde{t}} \frac{d\tilde{t}}{dt}, \ldots, \frac{1}{k!} \frac{d^{k-1}}{dt^{k-1}} \left(\frac{dx}{d\tilde{t}} \frac{d\tilde{t}}{dt}\right)\right)$$

The last equality holds for any diffeomorphism $\tilde{t} = \tilde{t}(t)$. Deriving it with respect to $d\tilde{t}/dt$ and taking $\tilde{t} = t$, we get

$$L = y^{(1)i} \frac{\partial L}{\partial y^{(1)i}} + \cdots + ky^{(k)i} \frac{\partial L}{\partial y^{(k)i}}$$

or $I^k(L) = L$. Deriving again (*) with respect to $d^2\tilde{t}/dt^2$ and taking $\tilde{t} = t$, we obtain $I^{k-1}(L) = 0$. By induction we have (2.8). QED

Kawaguchi (1961) and Kondo (1991) named equations (2.8) *Zermelo conditions*.

It is interesting to remark that if the Zermelo conditions hold, then $L(x,y^{(1)}, \ldots, y^{(k)})$ is not a regular Lagrangian (Kondo, 1991).

3. VARIATIONAL PROBLEM

Craig (1935) and Synge (1935) studied the variational problem for the integral of action $I(c)$ in (2.7). We add here some new considerations, which allow us to introduce important new operators useful in the proof of the Noether theorem.

Let c: $[0, 1] \to M$ be a smooth curve whose image belongs to the domain of a local chart $U \subset M$. Its extension to $\tilde{E}$ is c^*: $[0, 1] \to E$, given in (2.6).

On the open set $U \subset M$ we consider the curves c_ϵ: $[0, 1] \to M$:

$$c_\epsilon: t \in [0, 1] \to (x^i(t) + \epsilon V^i(t)) \in M \tag{3.1}$$

where ϵ is a real number sufficiently small in absolute value such that Im $c_\epsilon \subset U$ and $V^i(x(t))$ [denoted $V^i(t)$] is a regular vector field on the curve c. We assume that all curves c_ϵ have the same endpoints $c(0)$ and $c(1)$ with the curve c and their osculator spaces of order $1, \ldots, k-1$ to be coincident at the points $c(0)$ and $c(1)$. Therefore, the vector field $V^i(t)$ satisfies the conditions

$$\begin{gathered} V^i(0) = V^i(1) = 0 \\ \frac{dV^i}{dt}(0) = \frac{dV^i}{dt}(1) = 0, \ldots, \\ \frac{d^{k-1}V^i}{dt^{k-1}}(0) = \frac{d^{k-1}V^i}{dt^{k-1}}(1) = 0 \end{gathered} \tag{3.2}$$

The integral of action $I(c_\epsilon)$ of the differentiable Lagrangian $L(x, y^{(1)}, \ldots, y^{(k)})$ is as follows:

$$I(c_\epsilon) = \int_0^1 L\left(x + \epsilon V, \frac{dx}{dt} + \epsilon \frac{dV}{dt}, \ldots, \frac{1}{k!}\left(\frac{d^k x^i}{dt^k} + \epsilon \frac{d^k V}{dt^k}\right)\right) dt$$

A necessary condition for $I(c)$ to be an extremal value for $I(c_\epsilon)$ is

$$\left.\frac{dI(c_\epsilon)}{d\epsilon}\right|_{\epsilon=0} = 0 \tag{*}$$

We have

$$\frac{dI(c_\epsilon)}{d\epsilon} = \int_0^1 \frac{d}{d\epsilon}\left(L\left(x + \epsilon V, \frac{dx}{dt} + \epsilon \frac{dV}{dt}, \ldots, \frac{1}{k!}\left(\frac{d^k x}{dt^k} + \epsilon \frac{d^k V}{dt^k}\right)\right)\right) dt$$

and the Taylor expansion of L at the point $\epsilon = 0$ gives

$$\left.\frac{dI(c_\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \int_0^1 \left(\frac{\partial L}{\partial x^i} V^i + \frac{\partial L}{\partial y^{(1)i}} \frac{dV^i}{dt} + \cdots + \frac{1}{k!} \frac{\partial L}{\partial y^{(k)i}} \frac{d^k V^i}{dt^k}\right) dt$$

Now, putting

$$\begin{cases} I_V^1(L) = V^i \dfrac{\partial L}{\partial y^{(k)i}}, \qquad I_V^2(L) = V^i \dfrac{\partial L}{\partial y^{(k-1)i}} + \dfrac{dV^i}{dt} \dfrac{\partial L}{\partial y^{(k)i}}, \ldots, \\ I_V^k(L) = V^i \dfrac{\partial L}{\partial y(1)^i} + \dfrac{dV^i}{dt} \dfrac{\partial L}{\partial y^{(2)i}} + \dfrac{1}{2!} \dfrac{d^2 V^i}{dt^2} \dfrac{\partial L}{\partial y^{(3)i}} + \cdots + \dfrac{1}{(k-1)!} \dfrac{d^{k-1} V^i}{dt^{k-1}} \dfrac{\partial L}{\partial y^{(k)i}} \end{cases} \tag{3.3}$$

and

$$\overset{0}{E}_i(L) = \frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial y^{(1)i}} + \cdots + (-1)^k \frac{1}{k!} \frac{d^k}{dt^k} \frac{\partial L}{\partial y^{(k)i}} \tag{3.4}$$

one deduces a very important identity:

$$\begin{aligned} &\frac{\partial L}{\partial x^i} V^i + \frac{\partial L}{\partial y^{(1)i}} \frac{dV^i}{dt} + \cdots + \frac{1}{k!} \frac{\partial L}{\partial y^{(k)i}} \frac{d^k V^i}{dt^k} \\ &\quad = \overset{0}{E}_i(L) V^i + \frac{d}{dt} I_V^k(L) - \frac{1}{2!} \frac{d^2}{dt^2} I_V^{k-1}(L) + \cdots + (-1)^{k-1} \frac{1}{k!} \frac{d^k}{dt^k} I_V^1(L) \end{aligned} \tag{3.5}$$

Also, using (3.2), we have

$$I_V^\alpha(L)(c(0)) = I_V^\alpha(L)(c(1)) = 0, \qquad \alpha = 1, \ldots, k \tag{**}$$

Consequently, we can write

$$\begin{aligned} \left.\frac{dI(c_\epsilon)}{d\epsilon}\right|_{\epsilon=0} &= \int_0^1 \overset{0}{E}_i(L) V^i\, dt + \int_0^1 \frac{d}{dt} \left\{ I_V^k(L) - \frac{1}{2!} \frac{d}{dt} I_V^{k-1}(L) \right. \\ &\quad \left. + \cdots + (-1)^{k-1} \frac{1}{k!} \frac{d^{k-1}}{dt^{k-1}} I_V^1(L) \right\} dt \end{aligned} \tag{3.6}$$

By means of (**) it follows that

$$\left.\frac{dI(c_\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \int_0^1 \overset{0}{E}_i(L) V^i\, dt \tag{3.6'}$$

Now, taking into account the fact that V^i is an arbitrary vector field, then (3.6′) and (*) lead to the following result:

Theorem 3.1. In order for the integral of action $I(c)$ to be an extremal value for $I(c_\epsilon)$ it is necessary that the following Euler–Lagrange equation hold:

$$\overset{0}{E}_i(L) \overset{\text{def}}{=} \frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial y^{(1)i}} + \cdots + (-1)^k \frac{1}{k!}\frac{d^k}{dt^k}\frac{\partial L}{\partial y^{(k)i}} = 0$$

$$y^{(1)i} = \frac{dx^i}{dt}, \ldots, y^{(k)i} = \frac{1}{k!}\frac{d^k x^i}{dt^k} \tag{3.7}$$

The curves c: $[0, 1] \to M$, solutions of equation (3.7), are called extremal curves of the integral of action $I(c)$.

The equality (3.6′) implies the following result:

Theorem 3.2. $\overset{0}{E}_i(L)$ is a covector field.

Proof. Under a coordinate transformation (1.1), from (3.6′) it follows that

$$\int_0^1 \left[\overset{0}{\tilde{E}}_i(\tilde{L})\tilde{V}^i - \overset{0}{E}_i(L)V^i\right] dt$$

$$= \int_0^1 \left[\overset{0}{\tilde{E}}_i(\tilde{L})\frac{\partial \tilde{x}^i}{\partial x^j} - \overset{0}{E}_j(L)\right] V^j\, dt = 0$$

But V^i is an arbitrary vector field. One deduces

$$\overset{0}{\tilde{E}}_i(\tilde{L})\frac{\partial \tilde{x}^i}{\partial x^j} = \overset{0}{E}_j(L) \quad \text{QED}$$

Remark. $\overset{0}{E}_i(L) = 0$ has a geometrical meaning.

4. OPERATORS d_V/dt, $I_V^1, \ldots, I_V^k$

On further examination of the identity (3.5) we can introduce some important operators frequently used in the theory of higher-order Lagrangians.

Let c: $t \in [0, 1] \to (x^i(t)) \in M$ be a smooth curve, c^* as in (2.6), its extensions to $E = \mathrm{Osc}^k M$, and $V^i(x(t))$ a differentiable vector field along c.

It is easy to see that we have:

Lemma 4.1. The mapping S_V: $c \to \mathrm{Osc}^k M$, defined by

$$\begin{cases} x^i = x^i(t), \qquad t \in [0, 1] \\ y^{(1)i} = V^i(x(t)), \qquad 2y^{(2)i} = \dfrac{dV^i}{dt}, \ldots, \qquad ky^{(k)i} = \dfrac{1}{(k-1)!}\dfrac{d^{k-1}V^i}{dt^{k-1}} \end{cases} \tag{4.1}$$

is a section of the projection π: $\mathrm{Osc}^k M \to M$ along the curve c.

Indeed, using (1.1), we get

$$\tilde{y}^{(1)i} = \frac{\partial \tilde{x}^i}{\partial x^j} y^{(1)j} = \frac{\partial \tilde{x}^i}{\partial x^j} V^j = \tilde{V}^i, \ldots,$$

$$k\tilde{y}^{(k)i} = \frac{1}{(k-1)!}\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial \tilde{x}^i}{\partial x^j} V^j\right) = \frac{1}{(k-1)!}\frac{d^{k-1}\tilde{V}^i}{dt^{k-1}}$$

Clearly, if $V^i = dx^i/dt$, then $S_{dx/dt}(c) = c^*$.

The identity (3.5) suggests that we introduce the following operator along the curve c:

$$\frac{d_V}{dt} = V^i \frac{\partial}{\partial x^i} + \frac{dV^i}{dt}\frac{\partial}{\partial y^{(1)i}} + \cdots + \frac{1}{k!}\frac{d^k V^i}{dt^k}\frac{\partial}{\partial y^{(k)i}} \tag{4.2}$$

The importance of this operator results from:

Theorem 4.1. The operator d_V/dt has the following properties:

1. d_V/dt is invariant with respect to the coordinate transformations (1.1).

2. For any differentiable Lagrangian $L(x, y^{(1)}, \ldots, y^{(k)})$, $d_V L/dt$ is a scalar field.

3. d_V/dt is a derivative operator, i.e.,

$$\frac{d_V}{dt}(L + L') = \frac{d_V L}{dt} + \frac{d_V L'}{dt}, \qquad \frac{d_V(aL)}{dt} = a\frac{d_V L}{dt}, \qquad a \in R$$

$$\frac{d_V}{dt}(L \cdot L') = \frac{d_V L}{dt} \cdot L' + L \cdot \frac{d_V L'}{dt} \tag{4.3}$$

4. If $V^i = dx^i/dt$, then

$$\frac{d_V L}{dt} = \frac{dL}{dt}$$

Proof. 1. Using (1.1) and (1.2) along of the section S_V from (4.1), we have

$$\frac{d_V}{dt} = y^{(1)i}\frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i}\frac{\partial}{\partial y^{(k-1)i}} + \frac{dy^{(k)i}}{dt}\frac{\partial}{\partial y^{(k)i}}$$
$$= \tilde{y}^{(1)i}\frac{\partial}{\partial \tilde{x}^i} + 2\tilde{y}^{(1)i}\frac{\partial}{\partial \tilde{y}^{(1)i}} + \cdots + k\tilde{y}^{(k)i}\frac{\partial}{\partial \tilde{y}^{(k-1)i}} + \frac{d\tilde{y}^{(k)i}}{dt}\frac{\partial}{\partial \tilde{y}^{(k)i}} \quad (4.4)$$

The last equality proves the invariance of the operator d_V/dt with respect to the transformations of coordinates on the manifold E.

2. From part 1 we deduce $d_V\tilde{L}/dt = d_VL/dt$ for any Lagrangian L.

3. The particular form (4.2) of the operator d_V/dt implies (4.3).

4. In the case $V^i = dx^i/dt$ and observing that along the curve c we have

$$y^{(1)i} = \frac{dx^i}{dt}, \ldots, \quad y^{(k)i} = \frac{1}{k!}\frac{d^kx^i}{dt^k}$$

$$\frac{dL}{dt} = \frac{\partial L}{\partial x^i}y^{(1)i} + 2\frac{\partial L}{\partial y^{(1)i}}y^{(2)i} + \cdots + k\frac{\partial L}{\partial y^{(k-1)i}}y^{(k)i} + \frac{\partial L}{\partial y^{(k)i}}\frac{dy^{(k)i}}{dt} \quad (4.5)$$

it follows that $d_VL/dt = dL/dt$ for $V^i = dx^i/dt$.

For these reasons, d_V/dt is called *the total derivative in the direction of the vector field* V^i.

Now, let us consider the operators

$$\begin{cases} I_V^1 = V^i\dfrac{\partial}{\partial y^{(k)i}}, \qquad I_V^2 = V^i\dfrac{\partial}{\partial y^{(k-1)i}} + \dfrac{dV^i}{dt}\dfrac{\partial}{\partial y^{(k)i}}, \ldots, \\ I_V^k = V^i\dfrac{\partial}{\partial y^{(1)i}} + \dfrac{dV^i}{dt}\dfrac{\partial}{\partial y^{(2)i}} + \cdots + \dfrac{1}{(k-1)!}\dfrac{d^{k-1}V^i}{dt^{k-1}}\dfrac{\partial}{\partial y^{(k)i}} \end{cases} \quad (4.6)$$

Similar to the previous theorem we can prove:

Theorem 4.2. The following properties hold:

1. $I_V^1, \ldots, I_V^k$ are vector fields along the curve c.
2. $I_V^k = J(d_V/dt)$, $I_V^{k-1} = J(I_V^k), \ldots, I_V^1 = J(I_V^2)$.
3. $I_V^1(L), \ldots, I_V^k(L)$ are the scalars (3.3).
4. If $V^i = dx^i/dt$, then $I_V^1, \ldots, I_V^k$ are the Liouville vector fields $\overset{(1)}{\Gamma}, \ldots, \overset{(k)}{\Gamma}$ along the curve c.

Finally, the identity (3.5) leads to the following theorem:

Theorem 4.3. Along a smooth curve c of the manifold M we have

$$\frac{d_V L}{dt} = V^i \overset{0}{E}_i(L) + \frac{d}{dt} I_V^k(L) - \frac{1}{2!}\frac{d^2}{dt^2} I_V^{k-1}(L) + \cdots + (-1)^{k-1}\frac{1}{k!}\frac{d^k}{dt^k} I_V^1(L) \tag{4.7}$$

Indeed, (3.5) and (4.2) have as consequence the formula (4.7).

Corollary 4.1. For any Lagrangian $L(x, y^{(1)}, \ldots, y^{(k)})$ along a curve c we have

$$\frac{dL}{dt} = \frac{dx^i}{dt} \overset{0}{E}_i(L) + \frac{d}{dt} I^k(L) - \frac{1}{2!}\frac{d^2}{dt^2} I^{k-1}(L) + \cdots + (-1)^{k-1}\frac{1}{k!}\frac{d^k}{dt^k} I^1(L) \tag{4.8}$$

Corollary 4.2. If c is a solution curve of the Euler–Lagrange equation $\overset{0}{E}_i(L) = 0$ and along c we have

$$I^k(L) - \frac{1}{2!}\frac{dI^{k-1}(L)}{dt} + \cdots + (-1)^{k-1}\frac{1}{k!}\frac{d^{k-1}I^1(L)}{dt^{k-1}} = \text{const}$$

then the Lagrangian L is constant along c.

5. CRAIG–SYNGE COVECTORS

To the covectors field $\overset{0}{E}_i(L)$ along a curve c we shall associate other k covectors fields $\overset{1}{E}_i(L), \ldots, \overset{k}{E}_i(L)$ introduced 60 years ago independently by Craig (1935) and Synge (1935). These fields are useful in the geometry of regular Lagrangians of order k (Miron and Atanasiu, 1994, n.d.-b).

Let us consider a smooth curve c: $[0, 1] \to M$ and along c the operators

$$\overset{0}{E}_i = \frac{\partial}{\partial x^i} - \frac{d}{dt}\frac{\partial}{\partial y^{(1)i}} + \cdots + (-1)^k \frac{1}{k!}\frac{d^k}{dt^k}\frac{\partial}{\partial y^{(k)i}}$$

$$\overset{1}{E}_i = \sum_{\alpha=1}^{k} (-1)^\alpha \frac{1}{\alpha!}\binom{\alpha}{\alpha-1}\frac{d^{\alpha-1}}{dt^{\alpha-1}}\frac{\partial}{\partial y^{(\alpha)i}}$$

$$\overset{2}{E}_i = \sum_{\alpha=2}^{k} (-1)^\alpha \frac{1}{\alpha!}\binom{\alpha}{\alpha-2}\frac{d^{\alpha-2}}{dt^{\alpha-2}}\frac{\partial}{\partial y^{(\alpha)i}}$$

$$\vdots$$

$$\overset{k}{E}_i = (-1)^k \frac{1}{k!}\frac{\partial}{\partial y^{(k)i}} \tag{5.1}$$

These operators act R-linearly over the R-linear space of Lagrangians $L(x, y^{(1)}, \ldots, y^{(k)})$. We shall prove that $\overset{\alpha}{E}_i(L)$ $(\alpha = 1, \ldots, k)$ are covector fields.

To this aim, we first prove:

Lemma 5.1. For any differentiable Lagrangian $L(x, y^{(1)}, \ldots, y^{(k)})$ and any differentiable function $\phi(t)$ along the curve c we have

$$\overset{0}{E}_i(\phi L) = \phi \overset{0}{E}_i(L) + \frac{d\phi}{dt} \overset{1}{E}_i(L) + \cdots + \frac{d^k\phi}{dt^k} \overset{k}{E}_i(L) \tag{5.2}$$

Indeed, from (5.1) we deduce

$$\overset{0}{E}_i(\phi L) = \frac{\partial(\phi L)}{\partial x^i} - \frac{d}{dt}\frac{\partial(\phi L)}{\partial y^{(1)i}} + \cdots + (-1)^k \frac{1}{k!}\frac{d^k}{dt^k}\frac{\partial(\phi L)}{\partial y^{(k)i}}$$

Noticing that

$$\frac{\partial(\phi L)}{\partial x^i} = \phi \frac{\partial L}{\partial x^i}, \qquad \frac{\partial(\phi L)}{\partial y^{(\alpha)i}} = \phi \frac{\partial L}{\partial y^{(\alpha)i}} \qquad (\alpha = 1, \ldots, k)$$

and applying the Leibniz rule for calculating

$$\frac{d^\beta}{dt^\beta}\left(\phi \frac{\partial L}{\partial y^{(\alpha)i}}\right)$$

we get the identity (5.2).

Now, we can prove the following result without difficulty:

Theorem 5.1. For any Lagrangian $L(x, y^{(1)}, \ldots, y^{(k)})$ along a smooth curve c, $\overset{1}{E}_i(L), \ldots, \overset{k}{E}_i(L)$ are covector fields.

Remarks. 1. If L is a regular Lagrangian of order k, then the covector field $\overset{k-1}{E_i}(L)$ determines a k-spray and a nonlinear connection which depend on the Lagrangian L only (Miron and Atanasiu, n.d.-b).

2. $\partial L/\partial y^{(k)i}$ is a covector field on E.

Lemma 5.2. If F is a Lagrangian of order k with the property $\partial F/\partial y^{(k)i} = 0$, then the following equations hold along a smooth curve c:

$$\begin{aligned}
\frac{\partial}{\partial x^i}\frac{dF}{dt} &= \frac{d}{dt}\frac{\partial F}{\partial x^i} \\
\frac{\partial}{\partial y^{(1)i}}\frac{dF}{dt} &= \frac{\partial F}{\partial x^i} + \frac{d}{dt}\frac{\partial F}{\partial y^{(1)i}} \\
&\vdots \\
\frac{\partial}{\partial y^{(k-1)i}}\frac{dF}{dt} &= (k-1)\frac{\partial F}{\partial y^{(k-2)i}} + \frac{d}{dt}\frac{\partial F}{\partial y^{(k-1)i}} \\
\frac{\partial}{\partial y^{(k)i}}\frac{dF}{dt} &= k\frac{\partial F}{\partial y^{(k-1)i}}
\end{aligned} \tag{5.3}$$

Now we are able to prove an important result:

Theorem 5.2. For any differentiable Lagrangians $L(x, y^{(1)}, \ldots, y^{(k)})$, $F(x, y^{(1)}, \ldots, y^{(k-1)})$, $(\partial F/\partial y^{(k)i} = 0)$, along a smooth curve c we have

$$\overset{0}{E}_i\left(L + \frac{dF}{dt}\right) = \overset{0}{E}_i(L) \tag{5.4}$$

Proof. Taking into account the property

$$\overset{0}{E}_i\left(L + \frac{dF}{dt}\right) = \overset{0}{E}_i(L) + \overset{0}{E}_i\left(\frac{dF}{dt}\right)$$

and using (5.3), we get $\overset{0}{E}_i(dF/dt) = 0$. Therefore (5.4) holds. QED

Theorem 5.3. For any differentiable Lagrangian $F(x, y^{(1)}, \ldots, y^{(k-1)})$, along a smooth curve c we have

$$\overset{0}{E}_i\left(\frac{dF}{dt}\right) = 0, \quad \overset{1}{E}_i\left(\frac{dF}{dt}\right) = -\overset{0}{E}_i(F), \ldots, \quad \overset{k}{E}_i\left(\frac{dF}{dt}\right) = -\overset{k-1}{E}_i(F) \tag{5.5}$$

Corollary 5.1. If a differentiable Lagrangian F has the property $\partial F/\partial y^{(k)i} = 0$, then it also has the property

$$\overset{\alpha}{E}_i\left(\frac{dF}{dt}\right) = 0 \quad \text{implies} \quad \overset{\alpha-1}{E}_i(F) = 0 \quad (\alpha = 1, \ldots, k)$$

6. ENERGY $\mathscr{E}(L)$ AND ENERGIES OF ORDER $1, \ldots, K$, $\mathscr{E}_c^1(L), \ldots, \mathscr{E}_c^k(L)$

In the case $k = 1$ the notion of energy of a Lagrangian $L(x, y)$ is defined by $\mathscr{E}(L) = y^i \, \partial L/\partial y^i - L$. In terms of the main invariant $I^1(L) = y^i \, \partial L/\partial y^i$ the energy is expressed by $\mathscr{E}(L) = I^1(L) - L$. Therefore, it is natural to define the notion of *energy of a higher-order Lagrangian* $L(x, y^{(1)}, \ldots, y^{(k)})$ *by*

$$\mathscr{E}(L) = I^k(L) - L \tag{6.1}$$

or in a longer form

$$\mathscr{E}(L) = y^{(1)} \frac{\partial L}{\partial y^{(1)i}} + 2y^{(2)i} \frac{\partial L}{\partial y^{(2)i}} + \cdots + ky^{(k)i} \frac{\partial L}{\partial y^{(k)i}} - L \tag{6.2}$$

The function $\mathscr{E}(L)$ is a differentiable Lagrangian of order k.

For $k = 1$ along a smooth curve c we have

$$\frac{d\mathscr{E}(L)}{dt} = -\frac{dx^i}{dt} \overset{0}{E}_i(L) \tag{6.3}$$

where

$$\overset{0}{E}_i = \frac{\partial}{\partial x^i} - \frac{d}{dt}\frac{\partial}{\partial y^i}$$

If c is a solution curve of the Euler–Lagrange equation $\overset{0}{E}_i(L) = 0$, then $\mathscr{E}(L)$ is constant along the curve c. In the general case, for $k > 1$, this important result does not hold.

Namely, for $k > 1$ there exist some obstructions to the conservation of the energy $\mathscr{E}(L)$ along the solution curves of the Euler–Lagrange equation $\overset{0}{E}_i(L) = 0$.

We shall prove the existence of the mentioned obstructions.

Theorem 6.1. The energy $\mathscr{E}(L)$ of a differentiable Lagrangian $L(x, y^{(1)}, \dots, y^{(k)})$ is conserved along the solution curves c of the Euler–Lagrange equation $\overset{0}{E}_i(L) = 0$ if, and only if, along c we have

$$\frac{1}{2!}\frac{d}{dt}I^{k-1}(L) - \frac{1}{3!}\frac{d^2}{dt^2}I^{k-2}(L) + \cdots + (-1)^k\frac{1}{k!}\frac{d^{k-1}}{dt^{k-1}}I^1(L) = \text{const} \tag{6.4}$$

Proof. Let c be a smooth curve in the manifold M. From (6.1) we deduce

$$\frac{d\mathscr{E}(L)}{dt} = \frac{dI^k(L)}{dt} - \frac{dL}{dt}$$

By means of (4.8) it follows that

$$\frac{d\mathscr{E}(L)}{dt} = -\frac{dx^i}{dt}\overset{0}{E}_i(L) + \frac{1}{2!}\frac{d^2I^{k-1}}{dt^2} - \frac{1}{3!}\frac{d^3I^{k-2}}{dt^3} + \cdots + (-1)^k\frac{1}{k!}\frac{d^kI^1(L)}{dI^k} \tag{6.5}$$

Consequently, $\mathscr{E}(L)$ is conserved along the solution curves of the equation $\overset{0}{E}_i(L) = 0$ if and only if (6.4) holds. QED

Remark. If the Lagrangian L satisfies the Zermelo conditions (2.8), then its energy $\mathscr{E}(L)$ vanishes.

The last theorem shows that it could be useful to introduce another kind of energy which depends on the curve c, but is conserved along the solution curve c of the Euler–Lagrange equation (Leon *et al.*, 1985, 1992; Krupka, 1983, etc.).

Definition 6.1. We call energies of order $k, k - 1, \dots, 1$ of the Lagrangian $L(x, y^{(1)}, \dots, y^{(k)})$ with respect to a curve c the following invariants:

$$\begin{aligned}
\mathscr{E}_c^k(L) &= I^k(L) - \frac{1}{2!}\frac{dI^{k-1}(L)}{dt} + \cdots + (-1)^{k-1}\frac{1}{k!}\frac{d^{k-1}I^1(L)}{dt^{k-1}} - L \\
\mathscr{E}_c^{k-1}(L) &= -\frac{1}{2!}I^{k-1}(L) + \frac{1}{3!}\frac{dI^{k-2}(L)}{dt} + \cdots + (-1)^{k-1}\frac{1}{k!}\frac{d^{k-2}}{dt^{k-2}}I^1(L) \\
\mathscr{E}_c^{k-2}(L) &= \frac{1}{3!}I^{k-2}(L) - \frac{1}{4!}\frac{dI^{k-3}}{dt} + \cdots + (-1)^{k-1}\frac{1}{k!}\frac{d^{k-3}}{dt^{k-3}}I^1(L) \\
&\vdots \\
\mathscr{E}_c^1(L) &= (-1)^{k-1}\frac{1}{k!}I^1(L)
\end{aligned} \tag{6.6}$$

The dependence of these invariants on the curve c is obvious. A first result:

Proposition 6.1. If the Lagrangian L satisfies the Zermelo conditions (2.8), then all energies $\mathscr{E}_c^k(L), \ldots, \mathscr{E}_c^1(L)$ vanish.

Also, we have:

Proposition 6.2. The following identities hold:

$$\begin{aligned}
\mathscr{E}_c^k(L) - \frac{d}{dt}\mathscr{E}_c^{k-1}(L) &= \mathscr{E}(L) \\
\mathscr{E}_c^{k-1}(L) - \frac{d}{dt}\mathscr{E}_c^{k-2}(L) &= -\frac{1}{2!}I^{k-1}(L) \\
&\vdots \\
\mathscr{E}_c^2(L) - \frac{d}{dt}\mathscr{E}_c^1(L) &= (-1)^{k-2}\frac{1}{(k-1)!}I^2(L)
\end{aligned} \tag{6.7}$$

As we shall see, the energies $\mathscr{E}_c^k(L), \ldots, \mathscr{E}_c^1(L)$ are involved in a Noether theory of symmetries of the higher-order Lagrangians. With this end in view we state the following result:

Lemma 6.1. For any differentiable Lagrangian $L(x, y^{(1)}, \ldots, y^{(k)})$ and any differentiable function $\tau: M \to R$ along a smooth curve $c: [0, 1] \to M$, we have

$$\begin{aligned}
&\frac{d\tau}{dt}L - \left[\frac{d\tau}{dt}I^k(L) + \frac{1}{2!}\frac{d^2\tau}{dt^2}I^{k-1}(L) + \cdots + \frac{1}{k!}\frac{d^k\tau}{dt^k}I^1(L)\right] \\
&= \tau\frac{d\mathscr{E}_c^k(L)}{dt} + \frac{d}{dt}\left\{-\tau\mathscr{E}_c^k(L) + \frac{d\tau}{dt}\mathscr{E}_c^{k-1}(L) - \frac{d^2\tau}{dt^2}\mathscr{E}_c^{k-2}(L)\right. \\
&\quad \left. + \cdots + (-1)^k\frac{d^{k-1}\tau}{dt^{k-1}}\mathscr{E}_c^1(L)\right\}
\end{aligned} \tag{6.8}$$

Proof. The right-hand side of this equality, by means of (6.7), successively becomes

$$-\frac{d\tau}{dt}\left\{\mathscr{E}_c^k(L) - \frac{d\mathscr{E}_c^{k-1}(L)}{dt}\right\} + \frac{d^2\tau}{dt^2}\left\{\mathscr{E}_c^{k-1}(L) - \frac{d\mathscr{E}_c^{k-2}(L)}{dt}\right\} + \cdots$$
$$+ (-1)^{k-1}\frac{d^{k-1}\tau}{dt^{k-1}}\left\{\mathscr{E}_c^2(L) - \frac{d\mathscr{E}_c^1(L)}{dt}\right\} + (-1)^k \frac{d^k\tau}{dt^k}\mathscr{E}_c^1(L)$$
$$= -\frac{d\tau}{dt}\{I^k(L) - L\} - \frac{1}{2!}\frac{d^2\tau}{dt^2} I^{k-1}(L) - \frac{1}{3!}\frac{d^3\tau}{dt^3} I^{k-2}(L) - \cdots$$
$$- \frac{1}{(k-1)!}\frac{d^{k-1}\tau}{dt^{k-1}} I^2(L) - \frac{1}{k!}\frac{d^k\tau}{dt^k} I^1(L)$$
$$= \frac{d\tau}{dt}\cdot L - \left[\frac{d\tau}{dt} I^k(L) + \frac{1}{2!}\frac{d^2\tau}{dt^2} I^{k-1}(L) + \cdots + \frac{1}{k!}\frac{d^k\tau}{dt^k} I^1(L)\right]$$

Consequently, (6.8) holds. QED

An important result (Andreas *et al.*, 1991; Leon *et al.*, 1985) is given as follows:

Theorem 6.2. For any differentiable Lagrangian $L(x, y^{(1)}, \ldots, y^{(k)})$ along a smooth curve c: $[0, 1] \to (x^i(t)) \in M$, we have

$$\frac{d\mathscr{E}_c^k(L)}{dt} = -\overset{0}{E}_i(L)\frac{dx^i}{dt} \tag{6.9}$$

Indeed, from (6.6) we get

$$\frac{d\mathscr{E}_c^k(L)}{dt} = \frac{d}{dt}\left\{I^k(L) - \frac{1}{2!}\frac{dI^{k-1}(L)}{dt} + \cdots + (-1)^{k-1}\frac{1}{k!}\frac{d^{k-1}I^1(L)}{dt^{k-1}}\right\} - \frac{dL}{dt}$$

Substituting here dL/dt from (4.8) and performing the obvious reductions, we get (6.9).

An immediate consequence of the last theorem is the following:

Theorem 6.3. For any differentiable Lagrangian $L(x, y^{(1)}, \ldots, y^{(k)})$ the energy of order k, $\mathscr{E}_k^c(L)$, is conserved along of every solution curve c of the Euler–Lagrange equation $\overset{0}{E}_i(L) = 0$.

7. NOETHER THEOREMS

By Theorem 5.2, the integral of action $I(c)$, (2.7), of the differentiable Lagrangian $L(x, y^{(1)}, \ldots, y^{(k)})$ and the integral of action

$$I'(c) = \int_0^1 \left\{L\left(x, \frac{dx}{dt}, \ldots, \frac{1}{k!}\frac{d^kx}{dt^k}\right) + \frac{d}{dt}F\left(x, \frac{dx}{dt}, \ldots, \frac{1}{(k-1)!}\frac{d^{k-1}x}{dt^{k-1}}\right)\right\} dt$$

for any function $F(x, y^{(1)}, \ldots, y^{(k-1)})$, give rise to the same Euler–Lagrange equation $\overset{0}{E}_i(L)$, depending on the Lagrangian L only. Therefore, we can formulate:

Definition 7.1. A symmetry of the differentiable Lagrangian $L(x, y^{(1)}, \ldots, y^{(k)})$ is a C^∞-diffeomorphism $\varphi: R \times M \to R \times M$, which preserves the integral of action $I(c)$ of L.

For us it is very convenient to study the infinitesimal symmetries of higher-order Lagrangians. We start with an infinitesimal transformation on $R \times M$, given in the form

$$x'^i = x^i + \epsilon V^i(x, t) \qquad (i = 1, \ldots, n)$$
$$t' = t + \epsilon\tau(x, t) \tag{7.1}$$

where ϵ is a real number sufficiently small in absolute value such that the points (x, t) and (x', t') belong to the same local chart. Let c be the curve $c: t \in [0, 1] \to (t, x^i(t)) \in R \times M$. Terms of order greater than 1 in ϵ are neglected.

The inverse transformation of (7.1) is

$$x^i = x'^i - \epsilon V^i(x, t), \qquad t = t' - \epsilon\tau(x, t)$$

Along the curve c, $V^i(x(t), t)$ is a vector field. Applying Lemma 4.1, we find that S_V in (4.1) is a section in $\mathrm{Osc}^k M$ along c.

At the endpoints $c(0)$ and $c(1)$, V^i satisfies the conditions (3.2).

The infinitesimal transformation (7.1) is a symmetry for the Lagrangian $L(x, y^{(1)}, \ldots, y^{(k)})$ if and only if for any C^∞-function $F(x, y^{(1)}, \ldots, y^{(k-1)})$ the following equation holds:

$$L\left(x', \frac{dx'}{dt'}, \ldots, \frac{1}{k!}\frac{d^k x'}{dt'^k}\right) dt'$$
$$= \left\{L\left(x, \frac{dx}{dt}, \ldots, \frac{1}{k!}\frac{d^k x}{dt^k}\right) + \frac{dF}{dt}\left(x, \frac{dx}{dt}, \ldots, \frac{1}{(k-1)!}\frac{d^{k-1}x}{dt^{k-1}}\right)\right\} dt \tag{7.2}$$

From (7.1) we deduce

$$\frac{dt'}{dt} = 1 + \epsilon\frac{d\tau}{dt}$$
$$\frac{dx'^i}{dt'} = \frac{dx^i}{dt} + \epsilon\varphi^{(1)i}$$

$$\frac{1}{2!}\frac{d^2x'^i}{dt'^2} = \left(\frac{d^2x^i}{dt^2} + \epsilon\varphi^{(2)i}\right), \ldots,$$

$$\frac{1}{k!}\frac{d^kx'^i}{dt'^k} = \frac{1}{k!}\left(\frac{d^kx^i}{dt^k} + \epsilon\varphi^{(k)i}\right) \tag{7.3}$$

where we have put

$$\varphi^{(1)i} = \frac{dV^i}{dt} - \frac{dx^i}{dt}\frac{d\tau}{dt}$$

$$\varphi^{(2)i} = \frac{d^2V^i}{dt^2} - \binom{2}{1}\frac{d^2x^i}{dt^2}\frac{d\tau}{dt} - \binom{2}{2}\frac{dx^i}{dt}\frac{d^2\tau}{dt^2}$$

$$\vdots$$

$$\varphi^{(k)i} = \frac{d^kV^i}{dt^k} - \binom{k}{1}\frac{d^kx^i}{dt^k}\frac{d\tau}{dt} - \binom{k}{2}\frac{d^{k-1}x^i}{dt^{k-1}}\frac{d^2\tau}{dt^2} - \cdots - \binom{k}{k}\frac{dx^i}{dt}\frac{d^k\tau}{dt^k} \tag{7.4}$$

By virtue of (7.3) and (7.4), the equality (7.2), neglecting the terms in $\epsilon^2, \epsilon^3, \ldots$, and putting $\phi = \epsilon F$, leads to

$$L\frac{d\tau}{dt} + \frac{\partial L}{\partial x^i}V^i + \frac{\partial L}{\partial y^{(1)i}}\varphi^{(1)i} + \cdots + \frac{1}{k!}\frac{\partial L}{\partial y^{(k)i}}\varphi^{(k)i} = \frac{d\phi}{dt} \tag{7.5}$$

Conversely, if (7.5) holds, for L, V^i, τ, and c given, then putting $\epsilon\phi(x, y^{(1)}, \ldots, y^{(k-1)}) = F(x, y^{(1)}, \ldots, y^{(k-1)})$ the equality (7.2) is satisfied for the infinitesimal transformation (7.1) neglecting terms of order ≥ 2 in ϵ.

But $\varphi^{(1)i}, \ldots, \varphi^{(k)i}$ are given by (7.4). It follows that the equality (7.5) is equivalent to

$$V^i\frac{\partial L}{\partial x^i} + \frac{dV^i}{dt}\frac{\partial L}{\partial y^{(1)i}} + \cdots + \frac{1}{k!}\frac{d^kV^i}{dt^k}\frac{\partial L}{\partial y^{(k)i}}$$

$$+ \left\{L\frac{d\tau}{dt} - \left[I^k(L)\frac{d\tau}{dt} + \frac{1}{2!}I^{k-1}(L)\frac{d^2\tau}{dt^2} + \cdots + \frac{1}{k!}I^1(L)\frac{d^k\tau}{dt^k}\right]\right\} = \frac{d\phi}{dt} \tag{7.6}$$

Using the operator (4.2), we can state the following result.

Theorem 7.1. A necessary and sufficient condition that an infinitesimal transformation (7.1) be a symmetry for the Lagrangian $L(x, y^{(1)}, \ldots, y^{(k)})$

along the smooth curve c is that the left-hand side of the equality

$$\frac{d_V L}{dt} + \left\{ L \frac{d\tau}{dt} - \left[I^k(L) \frac{d\tau}{dt} + \frac{1}{2!} I^{k-1}(L) \frac{d^2\tau}{dt^2} + \cdots + \frac{1}{k!} I^1(L) \frac{d^k\tau}{dt^k} \right] \right\} = \frac{d\phi}{dt} \tag{7.7}$$

be of the form $(d/dt)\phi(x, y^{(1)}, \ldots, y^{(k-1)})$ along c.

Theorem 4.3 and Lemma 5.1 show that (7.7) is equivalent to

$$V^i \overset{0}{E}_i(L) + \frac{d}{dt} I_V^k(L) - \frac{1}{2!} \frac{d^2}{dt^2} I_V^{k-1}(L) + \cdots + (-1)^{k-1} \frac{1}{k!} \frac{d^k}{dt^k} I_V^1(L)$$
$$+ \tau \frac{d}{dt} \mathscr{E}_c^k(L) + \frac{d}{dt} \left[-\tau \mathscr{E}_c^k(L) + \frac{d\tau}{dt} \mathscr{E}_c^{k-1}(L) - \cdots + (-1)^{k-1} \frac{d^{k-1}\tau}{dt^{k-1}} \mathscr{E}_c^1(L) \right] = \frac{d\phi}{dt} \tag{7.8}$$

By Theorem 6.3, $\overset{0}{E}_i(L) = 0$ implies $d\mathscr{E}_c^k(L)/dt = 0$ and (7.8) leads to the Noether theorem:

Theorem 7.2. For any infinitesimal symmetry (7.1) [which satisfies (7.7)] of a Lagrangian $L(x, y^{(1)}, \ldots, y^{(k)})$ and for any function $\phi(x, y^{(1)}, \ldots, y^{(k-1)})$, the function

$$\mathscr{F}^k(L, \phi) \overset{\text{def}}{=} I_V^k(L) - \frac{1}{2!} \frac{d}{dt} I_V^{k-1}(L) + \cdots + (-1)^{k-1} \frac{1}{k!} \frac{d^{k-1}}{dt^{k-1}} I_V^1(L)$$
$$- \tau \mathscr{E}_c^k(L) + \frac{d\tau}{dt} \mathscr{E}_c^{k-1}(L) - \cdots + (-1)^{k-1} \frac{d^{k-1}\tau}{dt^{k-1}} \mathscr{E}_c^1(L) - \phi \tag{7.9}$$

is conserved along the solution curves of the Euler–Lagrange equation $\overset{0}{E}_i(L) = 0$.

The functions $\mathscr{F}^k(L, \phi)$ in (7.9) contain the relative invariants $I_V^1(L)$, $\ldots$, $I_V^k(L)$, the energies of order $1, 2, \ldots, k$, $\mathscr{E}_c^1(L), \ldots, \mathscr{E}_c^k(L)$, and the function $\phi(x, y^{(1)}, \ldots, y^{(k-1)})$. It seems that $\mathscr{F}^k(L, \phi)$ are convenient for higher-order mechanics.

In particular, if the Zermelo conditions (2.8) are satisfied, then the energies $\mathscr{E}_c^1(L), \ldots, \mathscr{E}_c^k(L)$ vanish and we have a shorter form of the Noether theorem:

Theorem 7.3. For any infinitesimal symmetry (7.1) of a differentiable Lagrangian $L(x, y^{(1)}, \ldots, y^{(k)})$ which satisfies the Zermelo conditions (2.8) and for any differentiable functions $\phi(x, y^{(1)}, \ldots, y^{(k-1)})$ the function

$$\mathcal{F}^k(L, \phi) \overset{\text{def}}{=} I_V^k(L) - \frac{1}{2!}\frac{d}{dt} I_V^{k-1}(L) + \cdots + (-1)^{k-1}\frac{1}{k!}\frac{d^{k-1}}{dt^{k-1}} I_V^1(L) - \phi$$

is conserved along the solution curves of the Euler–Lagrange equation $\overset{0}{E}_i(L) = 0$.

In the case $k = 1$, the function $\mathcal{F}^1(L, \phi)$ reduces to

$$\mathcal{F}^1(L, \phi) \overset{\text{def}}{=} V^i \frac{\partial L}{\partial y^i} - \tau\left(y^i \frac{\partial L}{\partial y^i} - L\right) - \phi(x)$$

and Theorem 7.2 is the classical Noether theorem (Souriau, 1970).

If the order of k is 2, we have the function

$$\mathcal{F}^2(L, \phi) \overset{\text{def}}{=} I_V^2(L) - \frac{1}{2!}\frac{d}{dt} I_V^1(L) - \tau\mathscr{E}_c^2(L) + \frac{d\tau}{dt}\mathscr{E}_c^1(L) - \phi$$

where

$$I_V^1(L) = V^i \frac{\partial L}{\partial y^{(2)i}}, \qquad I_V^2 = V^i \frac{\partial L}{\partial y^{(1)i}} + \frac{dV^i}{dt}\frac{\partial L}{\partial y^{(2)i}}$$

$$\mathscr{E}_c^2(L) = I^2(L) - \frac{1}{2}\frac{dI^1(L)}{dt} - L, \qquad \mathscr{E}_c^1(L) = -\frac{1}{2} I^1(L)$$

Good applications can be found for Lagrangians of the form (2.3) in the higher-order electrodynamics.

The above theory can be extended to time-dependent higher-order Lagrangians.

8. CONCLUSIONS

In the present paper we have studied the extension to the k-velocity manifold of classical Lagrangian mechanics.

Our investigation has focused on the variational problem for the integral of action $I(c)$ of a higher-order Lagrangian $L(x, y^{(1)}, \ldots, y^{(k)})$. After finding the Zermelo conditions under which $I(c)$ does not depend on the parametrization of the curve c, applying classical methods from the variational calculus, we deduce the Euler–Lagrange equation $\overset{0}{E}_i(L) = 0$.

Some important new operators d_V/dt, $I_V^1, \ldots, I_V^k$ and new higher-order energies $\mathscr{E}_c^1(L), \ldots, \mathscr{E}_c^k(L)$ have appeared. This happens since the known concept of energy $\mathscr{E}(L)$ is not sufficient. Indeed, we showed that for $k > 1$ there are some obstructions to the conservation of the energy $\mathscr{E}(L)$ along the solution curves of the equation $\overset{0}{E}_i(L) = 0$. It is remarkable that the energy of order k, $\mathscr{E}_c^k(L)$ satisfies the equation

$$\frac{d\mathscr{E}_c^k(L)}{dt} = -\overset{0}{E}_i(L)\,\frac{dx^i}{dt}$$

Consequently, $\mathscr{E}_c^k(L)$ has the above conservation property.

The main part of this theory is subordinated to the aim of providing a Noether theorem. So we defined the notion of symmetry in Section 7 and we proved the Noether theorem, Theorem 7.2, and a shorter form of it that holds when the Zermelo conditions are satisfied. The invariants of the infinitesimal symmetries are explicitly written.

So, we have shown that the higher-order Lagrangian mechanics in the k-velocity space is a natural extension of classical Lagrangian mechanics.

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